Journal of Engineering Physics and Thermophysics, Vol. 76, No. 5, 2003

SOLUTION OF THE UNSTEADY HEAT-CONDUCTION EQUATION FOR A SYSTEM OF TWO BOUNDED HETEROGENEOUS CYLINDERS WITH THE USE OF THE INTEGRAL TRANSFORMATIONS OF HANKEL AND LAPLACE

A. V. Alifanov and V. M. Golub

UDC 536.24

Consideration has been given to the problem of uniform unsteady lateral heating of two bounded cylinders having dissimilar thermophysical characteristics and being in ideal thermal contact. The exact analytical solution for determination of a three-dimensional temperature field in real space has been found with the use of the integral-transformation method. The distinctive features of the solution obtained have been investigated and examples of specific calculations have been given.

Let us consider a system of ideal contact of two bounded cylinders with dissimilar thermophysical characteristics and a zero initial temperature in the plane $z = 0$. The cylinders have the same radius R and lengths l_1 and l_2 respectively. A heat source of constant surface strength Q_R begins to act throughout the lateral surface of the cylinders ($-l_1 \le z \le l_2$, $r = R$) at the initial instant of time. On the cylinders' ends, we have heat exchange with a zero-temperature ambient medium according to the Newton law with heat-transfer coefficients α_1 and α_2. It is necessary to find the distribution of the temperature field in the system at any instant of time. Mathematically the problem formulated has the form of two differential heat-conduction equations in cylindrical coordinates

$$\frac{\partial T_1}{\partial t} - a_1\left(\frac{\partial^2 T_1}{\partial z^2} + \frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r}\right) = 0\,, \quad -l_1 < z < 0\,; \quad \frac{\partial T_2}{\partial t} - a_2\left(\frac{\partial^2 T_2}{\partial z^2} + \frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r}\right) = 0\,, \quad 0 < z < l_2\,, \tag{1}$$

with the following initial and boundary conditions:

$$T_1(r, z, 0) = T_2(r, z, 0)\,, \quad \lambda_1 \frac{\partial T_1(r, -l_1, t)}{\partial z} = \alpha_1 T_1(r, -l_1, t)\,, \quad \lambda_2 \frac{\partial T_2(r, -l_2, t)}{\partial z} = -\alpha_2 T_2(r, l_2, t)\,; \tag{2}$$

$$\lambda_1 \frac{\partial T_1(R, z, t)}{\partial r} = \lambda_2 \frac{\partial T_2(R, z, t)}{\partial r} = Q_R\,, \quad \frac{\partial T_1(0, z, t)}{\partial r} = \frac{\partial T_2(0, z, t)}{\partial r} = 0\,. \tag{3}$$

In the region of contact ($z = 0$), let it be necessary to satisfy the conjugation conditions

$$T_1(r, 0, t) = T_2(r, 0, t)\,, \quad \lambda_1 \frac{\partial T_1(r, 0, t)}{\partial z} = \lambda_2 \frac{\partial T_2(r, 0, t)}{\partial z}\,. \tag{4}$$

To problem (1)–(4), we successively apply the finite integral Hankel transformation

$$\overline{T}_i(\mu_m, z, t) = \int_0^R r J_0\left(\mu_m \frac{r}{R}\right) T_i(r, z, t)\, dr$$

Physical-Technical Institute of the National Academy of Sciences of Belarus, 10 Kuprevich Str., Minsk, 220141, Belarus; email: phti@ns.igsac.by. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 76, No. 5, pp. 131–137, September–October, 2003. Original article submitted October 17, 2002; revision submitted January 19, 2003.

1062-0125/03/7605-1111$25.00 ©2003 Plenum Publishing Corporation

and the Laplace transformation

$$\bar{\bar{T}}_i(\mu_m, z, s) = \int_0^\infty \exp(-st)\, \bar{T}_i(\mu_m, z, t)\, dt\,,$$

where μ_m are the roots of the equation $J_1(\mu) = 0$. After the indicated operations, the initial problem is reduced to the following problem:

$$\frac{d^2\bar{\bar{T}}_1}{dz^2} - \sigma_{1m}^2 \bar{\bar{T}}_1 = -\frac{RJ_0(\mu_m, Q_R)}{s\lambda_1}\,, \quad -l_1 < z < 0\,; \quad \frac{d^2\bar{\bar{T}}_2}{dz^2} - \sigma_{2m}^2 \bar{\bar{T}}_2 = -\frac{RJ_0(\mu_m, Q_R)}{s\lambda_2}\,, \quad 0 < z < l_2\,; \tag{5}$$

$$\lambda_1 \frac{d\bar{\bar{T}}_1(\mu_m, -l_1, s)}{dz} = \alpha_1 \bar{\bar{T}}_1(\mu_m, -l_1, s)\,, \quad \lambda_2 \frac{d\bar{\bar{T}}_2(\mu_m, l_2, s)}{dz} = -\alpha_2 \bar{\bar{T}}_2(\mu_m, l_2, s)\,; \tag{6}$$

$$\bar{\bar{T}}_1(\mu_m, 0, s) = \bar{\bar{T}}_2(\mu_m, 0, s)\,, \quad \lambda_1 \frac{d\bar{\bar{T}}_1(\mu_m, 0, s)}{dz} = \lambda_2 \frac{d\bar{\bar{T}}_2(\mu_m, 0, s)}{dz}\,, \tag{7}$$

where

$$\sigma_{1m} = \sqrt{\frac{s}{a_1} + \gamma_m^2}\,; \quad \sigma_{2m} = \sqrt{\frac{s}{a_2} + \gamma_m^2}\,; \quad \gamma_m = \frac{\mu_m}{R}\,.$$

Finally, problem (1)–(4) with partial differential equations has led us to the system of ordinary differential equations (5) with boundary conditions (6) and conjugation conditions (7). The general solutions of Eqs. (5) are written as follows:

$$\bar{\bar{T}}_1(\mu_m, z, s) = A_m \cosh \sigma_{1m} z + B_m \sinh \sigma_{1m} z + \frac{a_1 R J_0(\mu_m)\, Q_R}{\lambda_1 s\,(s + a_1 \gamma_m^2)}\,,$$

$$\bar{\bar{T}}_2(\mu_m, z, s) = C_m \cosh \sigma_{2m} z + D_m \sinh \sigma_{2m} z + \frac{a_2 R J_0(\mu_m)\, Q_R}{\lambda_2 s\,(s + a_2 \gamma_m^2)}\,.$$

After determination of the integration constants A_m, B_m, C_m, and D_m with the use of the boundary conditions and the conjugation conditions, we obtain the expressions for the temperatures of the contacting cylinders in the transform space:

$$\bar{\bar{T}}_1(\mu_m, z, s) = \frac{RQ_RJ_0(\mu_m)}{\lambda_1} \left\{ \frac{\sqrt{a_1 a_2}}{s\sigma_{2m} Z_m} \lambda_1 (\sigma_{2m} \sinh \sigma_{2m} l_2 + h_2 \cosh \sigma_{2m} l_2 - h_2)(\sigma_{1m} \cosh \sigma_{1m}(l_1 + z) + \right.$$

$$+ h_1 \sinh \sigma_{1m}(l_1 + z)) - \frac{\sqrt{a_1 a_2}}{s\sigma_{1m} Z_m} \left[\lambda_2 \frac{\sigma_{2m}}{\sigma_{1m}} (\sigma_{2m} \sinh \sigma_{2m} l_2 + h_2 \cosh \sigma_{2m} l_2)(\sigma_{1m} \cosh \sigma_{1m}(l_1 + z) + h_1 (\sinh \sigma_{1m}(l_1 + z) - \right.$$

$$\left.\left. - \sinh \sigma_{1m} z)) + \alpha_1 (\sigma_{2m} \cosh \sigma_{2m} l_2 + h_2 \sinh \sigma_{2m} l_2) \cosh \sigma_{1m} z \right] + \frac{a_1}{s\,(s + a_1 \gamma_m^2)} \right\}, \tag{8}$$

$$\bar{\bar{T}}_2(\mu_m, z, s) = \frac{RQ_RJ_0(\mu_m)}{\lambda_2} \left\{ \frac{\sqrt{a_1 a_2}}{s\sigma_{1m} Z_m} \lambda_2 (\sigma_{1m} \sinh \sigma_{1m} l_1 + h_1 \cosh \sigma_{1m} l_1 - h_1)(\sigma_{2m} \cosh \sigma_{2m}(l_2 - z) + \right.$$

$$+ h_2 \sinh \sigma_{2m} (l_2 - z)) - \frac{\sqrt{a_1 a_2}}{s\sigma_{2m} Z_m} \left[\lambda_1 \frac{\sigma_{1m}}{\sigma_{2m}} (\sigma_{1m} \sinh \sigma_{1m} l_1 + h_1 \cosh \sigma_{1m} l_1) (\sigma_{2m} \cosh \sigma_{2m} (l_2 - z) + h_2 (\sinh \sigma_{2m} (l_2 - z) - \right.$$

$$\left. - \sinh \sigma_{2m} z)) + \alpha_2 (\sigma_{1m} \cosh \sigma_{1m} l_1 + h_1 \sinh \sigma_{1m} l_1) \cosh \sigma_{2m} z \right] + \left. \frac{a_2}{s (s + a_2 \gamma_m^2)} \right\} , \qquad (9)$$

where

$$Z_m (\mu_m, s) = \sqrt{a_1 a_2} \, (\lambda_1 \sigma_{1m} (\sigma_{1m} \sinh \sigma_{1m} l_1 + h_1 \cosh \sigma_{1m} l_1) (\sigma_{2m} \cosh \sigma_{2m} l_2 + h_2 \sinh \sigma_{2m} l_2) +$$

$$+ \lambda_2 \sigma_{2m} (\sigma_{1m} \cosh \sigma_{1m} l_1 + h_1 \sinh \sigma_{1m} l_1) (\sigma_{2m} \sinh \sigma_{2m} l_2 + h_2 \cosh \sigma_{2m} l_2)) , \quad h_1 = \frac{\alpha_1}{\lambda_1} , \quad h_2 = \frac{\alpha_2}{\lambda_2} .$$

To obtain explicit expressions for the temperature fields we must perform the operations of inverse Laplace and Hankel transformations on Eqs. (8) and (9) [1–3]. In this connection, the question as to whether the original functions of the temperature of the solutions obtained exist is topical. It is common knowledge that not any function $f(s)$ in the space of Laplace transforms can be a transform of a certain function $f(t)$ in real space [2]. As the analysis of the solutions of (8) and (9) shows, they satisfy all the requirements of existence of original functions and corresponding inverse transforms. The only singularity exists solely at the boundary of contact ($z = 0$). But this singularity leads to the appearance of a certain generalized function in the final solution and it is no barrier to the procedure of inverse transformation. This singularity will be discussed somewhat below. In the process of inversion of the solutions of (8) and (9), we have employed the standard integrals of Laplace transformations [4] and the Vashchenko–Zakharenko theorem of expansion and the Borel theorem of multiplication of transforms [2]. In view of the cumbersomeness of these equations and accordingly the procedures of inverse transformation, we give only the final result, omitting the procedure of the mathematical computations themselves. The spatial temperature field in the system of bounded heterogeneous cylinders that are in ideal thermal contact and heated from the lateral surface by a heat flux of constant power Q_R is determined by the following expressions:

$$T_1 (r, z, t) = \frac{2Q_R}{\lambda_1 R} \left\{ \sum_{k=1}^{\infty} \frac{A_{k0} \cos \gamma_{k0} z + B_{k0} \sin \gamma_{k0} z}{a_1^2 \gamma_{k0}^4 Z_{k0}} (1 - \exp (- a_1 \gamma_{k0}^2 t)) + \sum_{m=1}^{\infty} \frac{J_0 \left(\frac{\mu_m}{R} r \right)}{\gamma_m^2 J_0 (\mu_m)} \times \right.$$

$$\times \left[\sum_{k=1}^{\infty} \frac{\beta_{km}}{(\gamma_{km}^2 + \gamma_m^2) Z_{km}} ((A_{km} \cos \gamma_{km} z + B_{km} \sin \gamma_{km} z) F (t) + (A'_{km} \cos \gamma_{km} z + B'_{km} \sin \gamma_{km} z) G (t)) + \right.$$

$$\left. \left. + (1 - \exp (- a_1 \gamma_m^2 t)) + \frac{\lambda_1 (1 - \exp (- a_2 \gamma_m^2 t)) - \lambda_2 (1 - \exp (- a_1 \gamma_m^2 t))}{\lambda_2 + \sqrt{\frac{a_2}{a_1}} \lambda_1} \eta (z) \right] \right\} , \qquad (10)$$

$$T_2 (r, z, t) = \frac{2Q_R}{\lambda_2 R} \left\{ \sum_{k=1}^{\infty} \frac{C_{k0} \cos q_{k0} z + D_{k0} \sin q_{k0} z}{a_1^2 \gamma_{k0}^4 Z_{k0}} (1 - \exp (- a_1 \gamma_{k0}^2 t)) + \sum_{m=1}^{\infty} \frac{J_0 \left(\frac{\mu_m}{R} r \right)}{\gamma_m^2 J_0 (\mu_m)} \times \right.$$

$$\times\left[\sum_{k=1}^{\infty}\frac{\gamma_{km}}{(\gamma_{km}^2+\gamma_m^2)\,Z_{km}}\left((C_{km}\cos q_{km}z+D_{km}\sin q_{km}z)\,F(t)+(C'_{km}\cos q_{km}z+D'_{km}\sin q_{km}z)\,G(t)\right)+\right.$$

$$\left.\left.+\,(1-\exp(-a_2\gamma_m^2t))+\frac{\lambda_2\,(1-\exp(-a_1\gamma_m^2t))-\lambda_1\,(1-\exp(-a_2\gamma_m^2t))}{\lambda_1+\sqrt{\dfrac{a_1}{a_2}}\,\lambda_2}\,\eta\,(z)\right]\right\}. \qquad (11)$$

Here

$$F(t)=\left(1-\frac{\gamma_{km}^2}{\beta_{km}^2}\exp(-a_2\,\gamma_m^2t)-\frac{\gamma_m^2}{\beta_{km}^2}\left(\exp(-a_2\,\gamma_m^2t)-\frac{a_2}{a_1}\exp(-a_1\,(\gamma_{km}^2+\gamma_m^2)\,t)\right)\right),$$

$$G(t)=\left(1-\exp(-a_1\,\gamma_m^2t)-\frac{\gamma_m^2}{\gamma_{km}^2}\,(\exp(-a_1\,\gamma_m^2t)-\exp(-a_1\,(\gamma_{km}^2+\gamma_m^2)\,t))\right),$$

$$Z_{km}=\left(\lambda_1\left(h_1\left(\frac{\gamma_{km}}{\beta_{km}}(1+h_2l_2)+\frac{\beta_{km}}{\gamma_{km}}\right)-\gamma_{km}\,\beta_{km}l_1\right)+\lambda_2\left(h_2\left(\frac{\beta_{km}}{\gamma_{km}}(1+h_1l_1)+\frac{\gamma_{km}}{\beta_{km}}\right)-\frac{a_1\,\gamma_{km}\beta_{km}l_2}{a_2}\right)\right)\times$$

$$\times\cos\gamma_{km}l_1\cos q_{km}l_2+\left(\lambda_1\left(\sqrt{\frac{a_2}{a_1}}\,h_2\left(\frac{h_1}{\gamma_{km}}-\gamma_{km}l_1\right)-\sqrt{\frac{a_1}{a_2}}\,\gamma_{km}h_1l_2\right)-\sqrt{\frac{a_1}{a_2}}\,\lambda_2\times\right.$$

$$\left.\times\left(\gamma_{km}\,(2+h_2l_2)+\frac{\beta_{km}^2}{\gamma_{km}}(1+h_1l_1)\right)\right)\cos\gamma_{km}l_1\sin q_{km}l_2+\left(\lambda_2\left(\frac{h_1h_2}{\beta_{km}}-\beta_{km}\left(h_2l_1+\frac{a_1}{a_2}h_1l_2\right)\right)-\right.$$

$$\left.-\lambda_1\left(\beta_{km}\,(2+h_1l_1)+\frac{\gamma_{km}^2}{\beta_{km}}(1+h_2l_2)\right)\right)\cos q_{km}l_2\sin\gamma_{km}l_1+\left(\lambda_1\left(\sqrt{\frac{a_1}{a_2}}\,\gamma_{km}^2l_2-\sqrt{\frac{a_2}{a_1}}\,h_2\,(2+h_1l_1)\right)+\right.$$

$$\left.+\sqrt{\frac{a_1}{a_2}}\,\lambda_2\,(\beta_{km}^2l_1-h_1\,(2+h_2l_2))\right)\sin\gamma_{km}l_1\sin q_{km}l_2\,,$$

$$A_{k0}=2\left\{(a_1\,\lambda_2-a_2\,\lambda_1)\,(\sqrt{a_1}\,\gamma_{k0}\cos\gamma_{k0}l_1+\sqrt{a_1}\,h_1\sin\gamma_{k0}l_1)\,(h_2\cos q_{k0}\,l_2-q_{k0}\sin q_{k0}l_2)+\right.$$

$$\left.+\,\alpha_1a_1\,(\sqrt{a_1}\,\gamma_{k0}\cos q_{k0}l_2+\sqrt{a_2}\,h_2\sin q_{k0}l_2)+h_2a_2\lambda_1\,(\sqrt{a_1}\,\gamma_{k0}\cos\gamma_{k0}l_1+\sqrt{a_1}\,h_1\sin\gamma_{k0}l_1)\right\},$$

$$B_{k0}=2\left\{(a_1\,\lambda_2-a_2\,\lambda_1)\,\sqrt{a_1}\,(h_1\cos\gamma_{k0}l_1-\gamma_{k0}\sin\gamma_{k0}l_1)\,(h_2\cos q_{k0}\,l_2-q_{k0}\sin q_{k0}l_2)-\right.$$

$$\left.-\,\alpha_1a_1\,\sqrt{a_2}\,(h_2\cos q_{k0}l_2-q_{k0}\sin q_{k0}l_2)+h_2a_2\lambda_1\,\sqrt{a_1}\,(h_1\cos\gamma_{k0}l_1-\gamma_{k0}\sin\gamma_{k0}l_1)\right\},$$

$$A_{km}=2\lambda_1\,(\gamma_{km}\cos\gamma_{km}l_1+h_1\sin\gamma_{km}l_1)\,(q_{km}\sin q_{km}l_2-h_2\cos q_{km}l_2+h_2)\,,$$

$$B_{km}=2\lambda_1\,(h_1\cos\gamma_{km}l_1-\gamma_{km}\sin\gamma_{km}l_1)\,(q_{km}\sin q_{km}l_2-h_2\cos q_{km}l_2+h_2)\,,$$

$$A'_{km}=2\left[\lambda_2\,(\gamma_{km}\cos\gamma_{km}l_1+h_1\sin\gamma_{km}l_1)\,(h_2\cos q_{km}l_2-q_{km}\sin q_{km}l_2)+\right.$$

$$+\sqrt{\frac{a_2}{a_1}}\,\frac{\alpha_1\,\gamma_{km}}{\beta_{km}}\,(q_{km}\cos q_{km}l_2 + h_2\sin q_{km}l_2)\Bigg],$$

$$B'_{km} = 2\lambda_2\,(h_2\cos q_{km}l_2 - q_{km}\sin q_{km}l_2)\,(h_1\cos\gamma_{km}l_1 - \gamma_{km}\sin\gamma_{km}l_1 + h_1)\,,$$

$$C_{k0} = 2\,\Big\{(a_2\,\lambda_1 - a_1\,\lambda_2)\,(h_1\cos\gamma_{k0}l_1 - \gamma_{k0}\sin\gamma_{k0}l_1)\,(\sqrt{a_1}\,\gamma_{k0}\cos q_{k0}\,l_2 + \sqrt{a_2}\,h_2\sin q_{k0}l_2) +$$

$$+\,\alpha_2 a_2\,(\sqrt{a_1}\,\gamma_{k0}\cos\gamma_{k0}l_1 + \sqrt{a_1}\,h_1\sin\gamma_{k0}l_1) + h_1 a_1\lambda_2\,(\sqrt{a_1}\,\gamma_{k0}\cos q_{k0}l_2 + \sqrt{a_2}\,h_2\sin q_{k0}l_2)\Big\}\,,$$

$$C_{km} = 2\sqrt{\frac{a_2}{a_1}}\Bigg[\lambda_1\,(h_1\cos\gamma_{km}l_1 - \gamma_{km}\sin\gamma_{km}l_1)\,(q_{km}\cos q_{km}l_2 + h_2\sin q_{km}l_2) +$$

$$+\sqrt{\frac{a_1}{a_2}}\,\frac{\alpha_2\,\beta_{km}}{\gamma_{km}}\,(\gamma_{km}\cos\gamma_{km}l_1 + h_1\sin\gamma_{km}l_1)\Bigg],$$

$$C'_{km} = 2\sqrt{\frac{a_2}{a_1}}\,\lambda_2\,(q_{km}\cos q_{km}l_2 + h_2\sin q_{km}l_2)\,(\gamma_{km}\sin\gamma_{km}l_1 - h_1\cos\gamma_{km}l_1 + h_1)\,,$$

$$D_{km} = \sqrt{\frac{a_2}{a_1}}\,B_{km}\,,\quad D'_{km} = \sqrt{\frac{a_2}{a_1}}\,B'_{km}\,,\quad \beta_{km} = \sqrt{\gamma_{km}^2 + \left(1 - \frac{a_2}{a_1}\right)\gamma_m^2}\,,\quad q_{km} = \sqrt{\frac{a_1}{a_2}}\,\beta_{km}\,,$$

$$\eta\,(z) = \begin{cases} 0 & \text{when} \quad z \neq 0\,, \\ 1 & \text{when} \quad z = 0\,. \end{cases}$$

The quantities γ_{km} are the roots of the transcendental equation

$$\lambda_1\,\gamma_{km}\,(\sqrt{a_1}\,\beta_{km}\cos q_{km}l_2 + \sqrt{a_2}\,h_2\sin q_{km}l_2)\,(h_1\cos\gamma_{km}l_1 - \gamma_{km}\sin\gamma_{km}l_1) +$$

$$+\,\lambda_2\sqrt{a_1}\,\beta_{km}\,(\gamma_{km}\cos\gamma_{km}l_1 + h_1\sin\gamma_{km}l_1)\,(h_2\cos q_{km}l_2 - q_{km}\sin q_{km}l_2) = 0\,. \qquad (12)$$

The analysis of Eqs. (10) and (11) obtained shows that they have a much more complex structure than the solutions for homogeneous cylinders [1, 5], one-dimensional problems of contact heat conduction [1, 5, 6], and three-dimensional problems in semibounded and unbounded regions [5–8]. In addition to the complex general form of the final expressions, we must focus our attention on two of their features.

The first feature is that in Eqs. (10) and (11) the indices of the exponents and the denominators of the terms of the sums of series involve the combination of the quantities $\gamma_{km}^2 + \gamma_m^2$. In the cases of homogeneous materials these terms are the roots of independent transcendental equations for the axial and radial directions. When the problems with bounded contacting bodies having dissimilar thermophysical characteristics are considered, the roots of longitudinal transcendental equations, conversely, depend on the quantities γ_m, which is obvious from expression (12). That is the reason why they have the double subscript km in our solution. We can call the occurring phenomenon the hybridization of the axial and radial roots. Let us assume that it is necessary to introduce m radial terms of the sum into consideration and k axial terms so as to attain the optimum exactness of the solution. Then in the case of a homogeneous material we will have to solve $m + k$ transcendental equations. If the problem is considered for two heterogeneous contacting bodies, this number increases to $m\,(k + 1)$. The analysis of Eq. (12) shows that it is precisely the dissimilar thermophysical characteristics of the contacting objects that are responsible for such a dependence. Indeed, if we set $\lambda_1 = \lambda_2 = \lambda$, $a_1 = a_2 = a$, $h_1 = h_2 = h$, $l_1 = 0$, and $l_2 = l$, we obtain the well-known ([1, 2, 5]) transcendental equation for a bounded homogeneous body with the same coefficients of heat transfer on the surfaces $z = 0$ and $z = l$:

TABLE 1. Roots of Eq. (12) for the Systems Copper–Titanium and Steel–Zirconium

m	μ_m	γ_{1m}		γ_{2m}		γ_{3m}		γ_{4m}		γ_{5m}	
		CuTi	FeZr	CuTi	FeZr	CuTi	FeZr	CuTi	FeZr	CuTi	FeZr
0	0	2.68	6.33	13.91	33.14	33.75	71.60	54.50	96.31	72.73	138.91
1	3.83	38.55	52.01	70.96	86.04	86.87	127.96	111.12	159.42	136.25	194.94
2	7.02	26.17	5.28	68.59	61.10	92.38	101.05	124.59	147.50	150.98	177.30
3	10.17	55.15	19.18	90.11	68.26	127.12	115.94	154.89	158.41	175.86	199.30
4	13.32	36.84	27.57	82.59	76.42	121.73	128.27	154.78	169.16	179.95	217.34
5	16.47	21.48	33.06	66.98	85.69	110.79	137.94	151.04	181.53	178.78	229.26

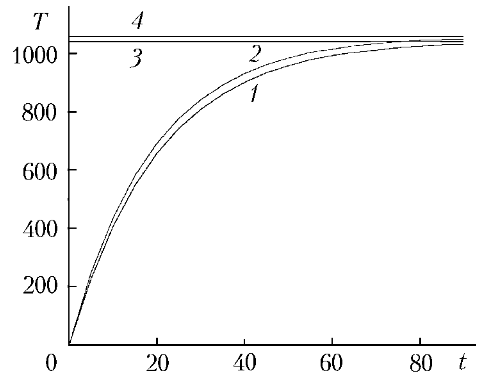

Fig. 1. Change in the temperature in the contact plane on the surface of the cylinders in the process of heating: 1) copper–titanium system; 2) steel–zirconium system; 3 and 4) steady-state temperatures calculated for these systems from the results of [9].

$$h\,(\gamma \cos \gamma l + h \sin \gamma l) + \gamma\,(h \cos \gamma l - \gamma \sin \gamma l) = 0\,.$$

The second feature of the solution obtained is in the presence of a point unit function $\eta(z)$ in it. Mathematically the appearance of this function is predetermined by the fact that expressions for the temperatures in the transform space (8) and (9) have finite limits at the point $z = 0$ on condition that $s \to \infty$: $\left(\lambda_2 + \sqrt{\dfrac{a_2}{a_1}}\,\lambda_1\right)^{-1}$ for $\bar{\bar{T}}_1(\mu_m, z, s)$ and $\left(\lambda_1 + \sqrt{\dfrac{a_1}{a_2}}\,\lambda_2\right)^{-1}$ for $\bar{\bar{T}}_2(\mu_m, z, s)$. This effect is absent when the contact one-dimensional [1, 5] and unbounded [8–8] problems of heat conduction are considered. It is also noteworthy that in the steady-state case the solution of our problem has no singularities in the contact plane and it is described by smooth functions [9].

In closing, we give results of calculations of the temperature fields in the process of diffusion welding of two different systems consisting of cylinders with dissimilar thermophysical characteristics (copper–titanium and steel–zirconium). To make the comparison more lucid we have selected the identical dimensions of the cylinders and parameters of heating (radii and lengths of the cylinders 40 mm, heat-transfer coefficients on all the ends 100 W/(m^2·deg), and heating power 1 kW/m^2). Thus, the dynamics of the spatial distribution of temperature will depend only on the characteristics of heated materials. Table 1 gives the first five roots of the equation $J_1(\mu) = 0$, which are related to the quantities γ_m by the relation $\mu_m = \gamma_m R$, and the corresponding roots of the transcendental equation (12) γ_{km}.

As is seen from the table, the quantities q_{km} strongly depend on both the roots of the radial equation $J_1(\mu) = 0$ (they change with μ_m within one system) and the form of the contacting materials (they are dissimilar for the same μ_m in different systems).

Figure 1 plots the temperature in the contact plane on the cylinders' surface as a function of time for the two systems. The same figure gives the values of the temperature in the steady state under the analogous conditions of

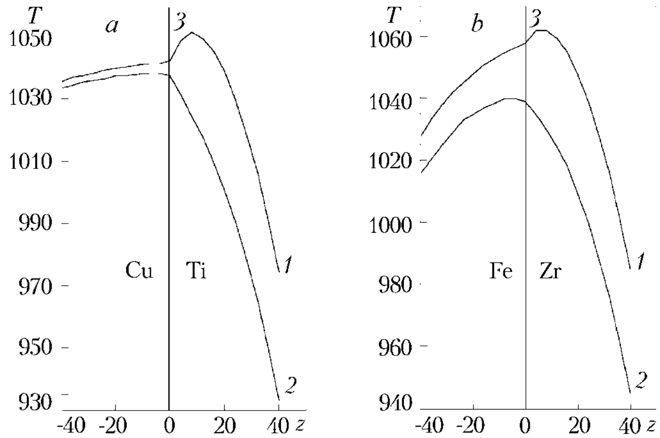

Fig. 2. Axial temperature distribution in the systems copper–titanium (a) and steel–zirconium (b): 1) along the generatrix of the cylinders; 2) along the central axis; 3) boundary of the contact plane.

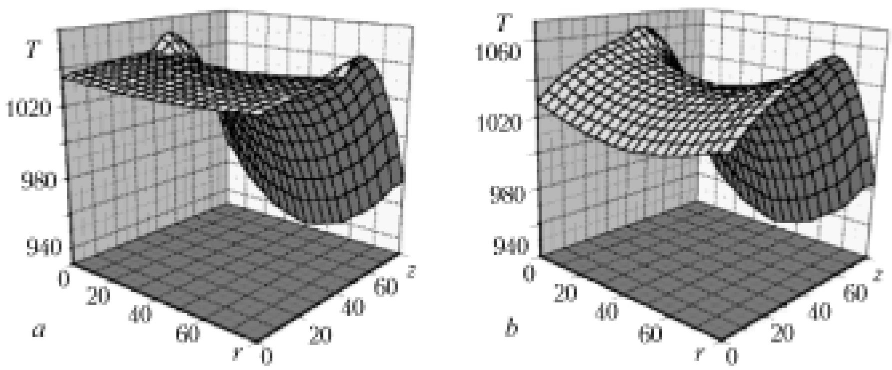

Fig. 3. Spatial distribution of the temperature field in the central plane of the cylinders: a) copper–titanium; b) steel–zirconium.

heating and heat transfer, calculated from the results of [9]. As follows from the plots, the calculation results (level of reaching the steady-state regime) are consistent with a satisfactory degree of accuracy.

The nearly coincident curves of change in the temperature in Fig. 1 can lead us to the conclusion that, under the same conditions of heat exchange, the distribution of the temperature field in dissimilar systems depens weakly on their thermophysical characteristics. But this is not quite the case. Figure 2 gives the longitudinal temperature distribution on the surface of the contacting cylinders and along their central axis in the nearly steady state. On the surface, the maximum is shifted from the interface (symmetry plane of the systems) toward the material with a lower coefficient of thermal diffusivity a (titanium or zirconium). On the central axis immediately behind the contact plane, the temperature rapidly decreases in the same direction. These effects are characteristic of the two systems but they are much more pronounced in the case of copper and titanium. We can also note an insignificant change in the temperature from the surface toward the center for the copper cylinder. The difference in the character of the temperature fields is more pronounced in Fig. 3, where their spatial distribution in the central cylinder planes is given. We observe a decrease in the temperature toward the central axis and the end surfaces which corresponds to boundary conditions (3) (lateral heating) and (2) (heat transfer through the ends). But the difference in the behavior of the temperature function for different materials is quite significant. This is particularly true of copper, whose heating is constant throughout the volume, in practice. The reason is the thermal properties of the materials under study, which vary in a rather wide range. For example, the thermal-diffusivity coefficients in the system steel–zirconium are rather close ($a_1/a_2 \approx 1.8$), and they differ more than ten times for copper and titanium ($a_1/a_2 \approx 12.6$). This is precisely the reason why the forms of the temperature field in Fig. 3 are dissimilar.

Of great practical interest is such a parameter as the time of reaching the steady-state regime by the process. It follows from Fig. 1 that this period is very significant (about 1 h). At the same time, as experience shows, the

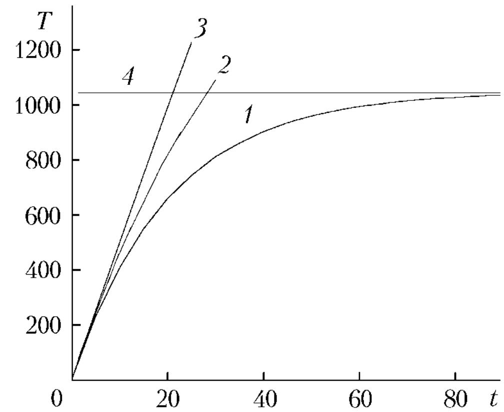

Fig. 4. Rate of change of the temperature in the contact plane vs. heat-transfer coefficient for the copper–titanium system: 1) α = 100, 2) 50, and 3) 5 W/(m^2·°C); 4) steady-state temperature calculated from the results of [9].

steady state is reached much more rapidly. This is attributable to the fact that, in vacuum heating, the basic mechanism of heat removal is radiation, and the heat-transfer coefficient α takes on low values at the beginning of the process since the temperature of the heated bodies differs little from the temperature of the ambient medium. The temperature of the system will increase much more rapidly. Figure 4 gives its change for different values of α. It is seen that for a small coefficient of heat transfer (characteristic of the beginning of the process) the operating regime is reached in a few minutes. This result is in good agreement with practical data.

Thus, the obtained analytical solution of the problem formulated enables one to adequately describe the spatial distribution of the temperature field and its dynamics in the system of bounded heterogeneous bodies for a constant coefficient of heat transfer and to quite accurately evaluate the process of heating with allowance for the change in the boundary conditions.

NOTATION

R, radius of the cylinders, m; l_1 and l_2, lengths of the cylinders, m; Q_R, surface power of the heat flux, W/m^2; α_1 and α_2, heat-transfer coefficients, W/(m^2·°C); a_1 and a_2, thermal-diffusivity coefficients, m^2/sec; λ_1 and λ_2, thermal-conductivity coefficients, W/(m^2·°C); h_1 and h_2, reduced coefficients of heat transfer, m^{-1}; T_1 and T_2, temperatures of the cylinders, °C; $\bar{\bar{T}}_1$ and $\bar{\bar{T}}_2$, temperatures of the cylinders in the transform domain, °C·m^2·sec; J_0 and J_1, Bessel functions of the first kind of zero and first orders; γ_{km}, roots of the transcendental equation (12); s, parameter of the Laplace transformation; t, time, sec; z and r, integration constants. Subscripts 1 and 2 refer to the first and second cylinders respectively.

REFERENCES

1. A. V. Luikov, *Theory of Heat Conduction* [in Russian], Moscow (1964).
2. É. M. Kartashov, *Analytical Methods in the Theory of Heat Conduction* [in Russian], Moscow (1985).
3. G. N. Watson, *A Treatise on the Theory of Bessel Functions* [Russian translation], Moscow (1949).
4. H. Bateman and A. Erdelyi, *Tables of Integral Transforms* [Russian translation], Vol. 1, Moscow (1969).
5. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* [Russian translation], Moscow (1961).
6. V. P. Kozlov, *Two-Dimensional Nonstationary Problems of Heat Conduction* [in Russian], Minsk (1986).
7. P. A. Mandrik and V. P. Kozlov, *Inzh.-Fiz. Zh.*, **74**, No. 2, 157–163 (2001).
8. P. A. Mandrik, *Inzh.-Fiz. Zh.*, **74**, No. 5, 153–159 (2001).
9. A. V. Alifanov and V. M. Golub, *Inzh.-Fiz. Zh.*, **76**, No. 1, 173–177 (2003).